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Two-dimensional s-equivalent Lagrangians and separability

Sergio Hojman and Sergio Ramos

Centro de Estudios Nucleares, Universidad Nacional Autónoma de México, Circuito Exterior, CU, 04510 México, DF

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Abstract. We prove that the necessary and sufficient condition for the existence of (non-trivial) Lagrangians s(olution)-equivalent to $L=\frac{1}{2}(\dot{x}^2+\dot{y}^2)-V(x,y,t)$ is that L can be written as $L=\frac{1}{2}(\dot{u}^2+\dot{v}^2)-V_1(u)-V_2(v)$ with $u=\alpha x+\beta y$, $v=\gamma x+\delta y$ where $\alpha,\beta,\gamma,\delta$ are constants.

1. Introduction

The inverse problem of the calculus of variations in two dimensions was solved by Douglas in 1941. He found the conditions the functions F and G had to satisfy in order that the equations

$$\ddot{x}=F(x,y,\dot{x},\dot{y},t) \qquad \ddot{y}=G(x,y,\dot{x},\dot{y},t) \tag{1a, b}$$

were equivalent to a variational problem. Douglas's work provides a complete classification of the possibilities which arise when studying the inverse problem of the calculus of variations associated with equations (1). Given the general solution to equations (1), the question is: does some action functional S exist such that the set of all its extremal curves coincides with the general solution to equations (1)? If so, is S unique? Note that this question is more general (and physically relevant) than the one asking: is there an action functional S such that its Euler–Lagrange equations coincide with equations (1)? In fact, no action functional reproducing equations (1) may exist, but there may be many of them giving rise to equations equivalent to equations (1) (see, for instance, Santilli 1978, Hojman and Shepley 1982). The inverse problem of the calculus of variations, then, deals with the existence and uniqueness of a Lagrangian function L

$$L=L(q^i,\dot{q}^i,t) \tag{2}$$

and the related action functional S

$$S=\int_{t_1}^{t_2} L\,\mathrm{d}t$$

such that the extremal curves $q^i=q^i(t)$ which imply

$$\delta S=0 \tag{3}$$

for arbitrary variations δq^i, $t_1 < t < t_2$ and

$$\delta q^i = 0 \qquad \text{for } t = t_1 \text{ and } t = t_2$$

coincide with the general solution of a given set of differential equations.

The inverse problem for one differential equation in one variable was solved by Darboux in 1891 and he proved that, in the one-dimensional case, there are always infinitely many Lagrangians for the general solution of a given differential equation.

For the general case much improvement was achieved in the recent work of Henneaux in 1981.

One closely related, although more restricted, subject is that of s(olution)-equivalent Lagrangians (Hojman and Shepley 1982, Currie and Saletan 1966, Hojman and Harleston 1981) which deals with the existence of Lagrangians non-trivially related to one which gives rise to the general solutions of a given set of differential equations.

It is worthwhile noting that while s-equivalent Lagrangians give rise to the same classical theory (as far as the general solutions of it are concerned), the quantum theories built up from them (using the canonical quantisation scheme, for instance) are not equivalent (Hojman and Shepley 1982, Marmo and Saletan 1978, Hojman and Montemayor 1980). Ambiguities also appear when considering the relationship between symmetries and conserved quantities in the case s-equivalent Lagrangians exist (Hojman and Shepley 1982, Marmo and Saletan 1977, Havas 1973).

We consider that the question of the existence of s-equivalent Lagrangians is a very important one and we deal with this problem in the case of two dimensions. In this work we prove that the necessary and sufficient condition for the existence of Lagrangians s-equivalent to

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V(x, y, t) \tag{4}$$

is that L can be written as

$$L = \tfrac{1}{2}(\dot{u}^2 + \dot{v}^2) - V_1(u, t) - V_2(v, t) \tag{5}$$

with

$$u = \alpha x + \beta y \qquad v = \gamma x + \delta y \tag{6a,b}$$

where α, β, γ and δ are constant.

In order to do this, in § 2 we present a short summary of Douglas' results. In § 3 we present the explicit proof of the statement just made. Section 4 contains some comments and conclusions.

2. Summary of the results obtained by Douglas

Consider the system of differential equations

$$\ddot{x} = F(x, y, \dot{x}, \dot{y}, t) \qquad \ddot{y} = G(x, y, \dot{x}, \dot{y}, t) \tag{1a, b}$$

and define

$$A_0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \dot{y}} - 2\frac{\partial F}{\partial y} - \frac{1}{2}\frac{\partial F}{\partial \dot{y}}\left(\frac{\partial F}{\partial \dot{x}} + \frac{\partial G}{\partial \dot{y}}\right) \tag{7a}$$

$$B_0 = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \dot{x}} + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial G}{\partial \dot{y}} + 2\left(\frac{\partial F}{\partial x} - \frac{\partial G}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial F}{\partial \dot{x}} - \frac{\partial G}{\partial \dot{y}}\right)\left(\frac{\partial F}{\partial \dot{x}} + \frac{\partial G}{\partial \dot{y}}\right) \tag{7b}$$

$$C_0 = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial G}{\partial \dot{x}}\right) + 2\frac{\partial G}{\partial \dot{x}} + \frac{1}{2}\,\frac{\partial G}{\partial x}\left(\frac{\partial F}{\partial \dot{x}} + \frac{\partial G}{\partial \dot{y}}\right) \tag{7c}$$

$$A_{i+1} = \frac{\mathrm{d}A_i}{\mathrm{d}t} - \frac{\partial F}{\partial \dot{x}} A_i - \frac{1}{2}\frac{\partial F}{\partial \dot{y}} B_i \tag{8a}$$

$$B_{i+1} = \frac{\mathrm{d}B_i}{\mathrm{d}t} - \frac{\partial G}{\partial \dot{x}} A_i - \frac{1}{2}\left(\frac{\partial F}{\partial \dot{x}} + \frac{\partial G}{\partial \dot{y}}\right) B_i - \frac{\partial F}{\partial \dot{y}} C_i \tag{8b}$$

$$C_{i+1} = \frac{\mathrm{d}C_i}{\mathrm{d}t} - \frac{1}{2}\,\frac{\partial G}{\partial \dot{x}} B_i - \frac{\partial G}{\partial \dot{y}} C_i \tag{8c}$$

for $i = 0, 1$ where

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + F\frac{\partial}{\partial \dot{x}} + G\frac{\partial}{\partial \dot{x}}. \tag{9}$$

The classification given by Douglas for the different cases which arise is the following.

Case I.

$$A_0 = 0 \qquad B_0 = 0 \qquad C_0 = 0 \tag{10a,b,c}$$

Case II. At least one of the following occur:

$$A_0 \neq 0 \qquad B_0 \neq 0 \qquad C_0 \neq 0$$

and

$$\Delta_1 \equiv A_0 B_1 - A_1 B_0 = 0 \qquad \Delta_2 \equiv B_0 C_1 - B_1 C_0 = 0 \qquad \Delta_3 \equiv C_0 A_1 - C_1 A_0 = 0 \tag{11a,b,c}$$

are true simultaneously.

Case III. At least one of the following occur:

$$\Delta_1 \neq 0 \qquad \Delta_2 \neq 0 \qquad \Delta_3 \neq 0$$

and

$$\Delta \equiv \begin{vmatrix} A_0 & B_0 & C_0 \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = 0. \tag{12}$$

Case IV.

$$\Delta \neq 0. \tag{13}$$

Given F and G the problem will be classified according to the scheme outlined above.

For cases I and II Lagrangian either exists and its is not unique or it does not exist. For cases III and IV a Lagrangian either exists and it is unique or it does not exist.

In the preceding sentences L and L' are considered the same if

$$L' = \rho L + \mathrm{d}f(q^i, t)/\mathrm{d}t \tag{14}$$

where ρ is a constant and f is an arbitrary function of q^i and t.

3. s-equivalence and separability

We now prove the statement made in the introduction. Given

$$L=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)-V(x,y,t) \tag{15}$$

the forces F and G are given by

$$F=-\partial V/\partial x \qquad G=-\partial V/\partial y. \tag{16a,b}$$

Due to the fact that one Lagrangian for the problem

$$\ddot{x}=F(x,y,t) \qquad \ddot{y}=G(x,y,t) \tag{1a,b}$$

already exists, namely L given by equation (15), only cases I and II will allow non-trivial equivalent Lagrangians to exist. (For cases III and IV when a Lagrangian exists, it is unique.)

Consider case I first. We have

$$A_0=0 \qquad B_0=0 \qquad C_0=0. \tag{10a,b,c}$$

For the forces given by equations (16)

$$A_0=2\partial^2V/\partial y\,\partial x \tag{17a}$$

$$B_0=-2(\partial^2V/\partial x^2-\partial^2V/\partial y^2) \tag{17b}$$

$$C_0=-2\partial^2V/\partial x\,\partial y=-A_0. \tag{17c}$$

In this situation equations (10a) and (10c) coincide. The conditions (10a) and (10b) read

$$\partial^2V/\partial x\,\partial y=0 \qquad \partial^2V/\partial x^2-\partial^2V/\partial y^2=0. \tag{18a,b}$$

The general solution of equation (18a) is

$$V=f(x,t)+g(y,t) \tag{19a}$$

where f and g are arbitrary functions of their arguments. Using the expression (19a) in (18b), one gets for V

$$V(x,y,t)=h(t)(x^2+y^2)+j_1(t)x+j_2(t)y+k(t) \tag{19b}$$

where h, j_1, j_2 and k are arbitrary functions of time. Therefore, V can be written as

$$V(x,y,t)=V_1(x,t)+V_2(y,t), \tag{20}$$

i.e. V is separable in the variables x and y.

For case II at least one of the following relations is satisfied:

$$A_0\neq 0, \qquad B_0\neq 0, \qquad C_0\neq 0 \tag{21a,b,c}$$

and

$$\Delta_1=0 \qquad \Delta_2=0 \qquad \Delta_3=0. \tag{22a,b,c}$$

Three cases arise:

(i) $A_0=-C_0\neq 0$ and $B_0=0$

(ii) $A_0=-C_0=0$ and $B_0\neq 0$

(iii) $A_0=-C_0\neq 0$ and $B_0\neq 0$.

In cases (i) and (ii) equations (22) are satisfied identically because

$$A_1 = \mathrm{d}A_0/\mathrm{d}t \qquad B_1 = \mathrm{d}B_0/\mathrm{d}t, \qquad C_1 = \mathrm{d}C_0/\mathrm{dt} = -A_1. \tag{23a,b,c}$$

Consider case (i), i.e. equation (18b)

$$\partial^2 V/\partial x^2 - \partial^2 V/\partial y^2 = 0. \tag{18b}$$

Its general solution is

$$V = V_+(x+y, t) + V_-(x-y, t). \tag{24}$$

Therefore, V is separable in the variables $u^1 = x+y, v^1 = x-y$ and the kinetic term in the Lagrangian takes the desired form for

$$u = (x+y)/\sqrt{2} \qquad v = (x-y)/\sqrt{2}. \tag{25a,b}$$

Therefore for case (i)

$$L = \tfrac{1}{2}\dot{u}^2 - V_1(u_1 t) + \tfrac{1}{2}\dot{v}^2 - V_2(v_1 t). \tag{26}$$

Case (ii) is even simpler. We have already proved with equations (18a) and (19a) that V is separable in the variables x and y.

Consider now case (iii). Equations (22a) and (22b) coincide while equation (22c) is identically satisfied due to equations (17c) and (23c).

We only need to consider equation (22a), namely

$$A_0\,\mathrm{d}B_0/\mathrm{d}t - B_0\,\mathrm{d}A_1/\mathrm{d}t = 0. \tag{27}$$

Due to the fact that both A_0 and B_0 are different from zero, equation (27) implies

$$\mathrm{d}/\mathrm{d}t(A_0/B_0) = 0 \tag{28}$$

i.e.

$$A_0/B_0 = 1/\rho = \text{constant} \tag{29}$$

but, due to fact that A_0 and B_0 are functions of x, y and t only (and not functions of $\dot{x}$ and $\dot{y}$), ρ has to be a numerical constant. Therefore

$$\partial^2 V/\partial x^2 - \partial^2 V/\partial y^2 - \rho\partial^2 V/\partial x\,\partial y = 0. \tag{30}$$

Consider the change of variables

$$u' = (\varepsilon x + y) \qquad v' = (x - \varepsilon y) \tag{31a,b}$$

where ε satisfies

$$\varepsilon^2 - \rho\varepsilon - 1 = 0. \tag{32}$$

Equation (30) becomes

$$(\rho\varepsilon^2 - 4\varepsilon - \rho)\partial^2 V/\partial u\,\partial v = 0 \tag{33}$$

and it is straightforward to prove that

$$(\rho\varepsilon^2 - 4\varepsilon - \rho) \neq 0 \tag{34}$$

for any ε which satisfies equation (32). Therefore,

$$V = V_1'(u', t) + V_2'(v', t) \tag{35}$$

and to get the desired form for the kinetic term we define

$$u = u'/(1+\varepsilon^2)^{1/2} \qquad v = v'/(1+\varepsilon^2)^{1/2} \tag{36a,b}$$

and L can be written as

$$L = \tfrac{1}{2}\dot{u}^2 - V_1(u, t) + \tfrac{1}{2}\dot{v}^2 - V_2(v, t). \tag{37}$$

We have, therefore, proved that L must have the form given by equations (37) with

$$u = \alpha x + \beta y \qquad v = \gamma x + \delta y \tag{6a,b}$$

when α, β, γ, δ are constants in order to possess non-trivial equivalent Lagrangians. Conversely, it is trivial to prove that if L can be written in the way given by equation (37) it has infinitely many equivalent Lagrangians (Hojman and Shepley 1982, Darboux 1891, Currie and Saletan 1966, Hojman and Harleston 1981).

4. Conclusions

We have proved that for the usual T-V Lagrangian the existence of other non-trivial Lagrangians s-equivalent to it is equivalent to separability in the two-dimensional case. This is not the case in three or more dimensions where there are counter examples (Henneaux and Shepley 1981). The study of the inverse problem of the calculus of variations remains therefore an interesting task especially in connection with the quantum theory and the relationship of symmetries and conservation laws (Hojman and Shepley 1982, Hojman and Montemayor 1980, Marmo and Saletan 1977, Hojman and Gómez 1982).

References

Currie D and Saletan E J 1966 *J. Math. Phys.* **7** 967
Darboux G 1891 *Leçons sur la Théorie Générale des Surfaces* III^e partie (Paris: Gauthier-Villars)
Douglas J 1941 *Trans. Am. Math. Soc.* **50** 71
Havas P 1973 *Acta Phys. Austriaca* **38** 145
Henneaux M 1981 *Equations of motion, Commutation Relations and Ambiguities in the Lagrangian Formalism, Université Libre de Bruxelles, Preprint*
Henneaux M and Shepley L C 1981 *Lagrangians for Spherically Symmetric Potentials, University of Texas at Austin, Preprint*
Hojman S and Gómez J 1982 *First Order Equivalent Lagrangians and Conservation Laws, Universidad Nacional Autónoma de México, Preprint*
Hojman S and Harleston H 1981 *J. Math. Phys.* **22** 1414
Hojman S and Montemayor R 1980 *Hadronic J.* **3** 1644
Hojman S and Shepley L C 1982 *Lagrangianos Equivalentes, Rev. Mex. Fís.* **28** 149
Marmo G and Saletan E J 1977 *Nuovo Cimento* B **40** 67
—— 1978 *Hadronic J.* **1** 955
Santilli R M 1978 *Foundations of Classical Mechanics I* (New York: Springer) p 110